Comparing Control Charts With Estimated Parameters

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We will look at

- Shewhart mean and variance charts
- Exponentially Weighted Moving Average (EWMA) chart for mean and Shewhart variance chart
- Cumulative Sum (CuSUM) chart for mean and Shewhart variance chart
Preliminaries

When parameters are known, at time $t$ we compute

$$Y_t = \frac{\sqrt{n}}{\sigma_0} (\bar{X}_t - \mu_0) \quad \text{and} \quad R_t = \frac{S_t^2}{\sigma_0^2}.$$
Letting
\[ \delta = \left( \frac{\mu - \mu_0}{\frac{\sigma_0}{\sqrt{n}}} \right) \]
and
\[ \gamma^2 = \frac{\sigma^2}{\sigma_0^2} \]

It turns out that
\[ Y_t \sim N(\delta, \gamma^2), \]
and
\[ \Pr[A < R_t < B] = \int_{(n-1)A/\gamma^2}^{(n-1)B/\gamma^2} h(x) \, dx, \]

where \( h(x) \) is the p.d.f. of the Chi-square distribution with \( n-1 \) degrees of freedom.
Preliminaries

When parameters are estimated, at time $t$ we compute

\[ Y_t = \frac{\sqrt{n}}{\hat{\sigma}_0} (X_t - \hat{\mu}_0) \quad \text{and} \quad R_t = \frac{S_t^2}{\hat{\sigma}_0^2}, \]

where the parameters are estimated from $m$ retrospective in-control samples of size $n$. 
$Y_t$ can be re-written as

$$Y_t = \frac{1}{W_0} (\gamma Z_t + \delta - \frac{Z_0}{\sqrt{m}}),$$

where $W_0 = \frac{\hat{\sigma}_0}{\sigma_0}$, $Z_0 = \frac{\hat{\mu}_0 - \mu_0}{\sigma_0 / \sqrt{mn}} \sim N(0,1)$, and $Z_t = \frac{\bar{X}_t - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$.

The conditional pdf of $Y_t$ given $W_0 = w_0$ and $Z_0 = z_0$

$$g_{y_t}(y_t \mid w_0, z_0) = \frac{w_0}{\gamma} \phi\left(\frac{w_0}{\gamma} y_t - \frac{\delta}{\gamma} + \frac{z_0}{\gamma \sqrt{m}}\right),$$

$$\Pr[A < R_t < B] = \int_{w_0^2(n-1)A/\gamma^2}^{w_0^2(n-1)B/\gamma^2} h(x)dx.$$
Joint $(\bar{X}, S^2)$ Chart

When parameters are known, the process is declared out-of-control at $t>0$ if either

$$|Y_t| \geq M \text{ or } (R_t \leq A \text{ or } R_t \geq B).$$

When parameters are estimated, the process is declared out-of-control at $t>0$ if either

$$|Y_t| \geq M \text{ or } (R_t \leq A \text{ or } R_t \geq B).$$
Joint \((\text{EWMA}, S^2)\) Chart

At time \(t>0\), we have computed \(Y_t\) and \(R_t\), if parameters are known and \(Y_t\) and \(R_t\), if parameters are estimated. We further compute the EWMA statistic

\[
Q_t = (1 - \lambda)Q_{t-1} + \lambda Y_t, \quad Q_0 = u.
\]

The process is declared out-of-control when

\[
|Q_t| \geq h \quad \text{or} \quad (R_t \leq A \quad \text{or} \quad R_t \geq B).
\]
At time $t>0$, we have computed $Y_t$ and $R_t$, if parameters are known and $Y_t$ and $R_t$, if parameters are estimated. We further compute

$$
C_t^+ = \max \left\{ 0, Y_t + C_{t-1}^+ - k \right\}
$$

$$
C_t^- = \max \left\{ 0, -Y_t - k + C_{t-1}^- \right\},
$$

$$
C_0^+ = u, \quad C_0^- = v, \quad 0 \leq u, v < h.
$$
Joint \((CUSUM, S^2)\) Chart

The process is declared out-of-control as soon as

\[ C_t^+ \geq h, \]
\[ C_t^- \geq h, \]

or \((R_t \leq A \text{ or } R_t \geq B)\).
We need equations for the ARL and SDRL

Joint \((\bar{X}, S^2)\) Chart

Joint \((EWMA, S^2)\) Chart

Joint \((CUSUM, S^2)\) Chart
ARL and SMRL for
(EWMA, S^2) chart with known parameters

\[ L(u, \delta, \gamma) = 1 + \frac{\Pr[A < R_1 < B]}{\lambda} \int_{-h}^{h} L(t, \delta, \gamma) f_{Y_t} \left( \frac{t - (1 - \lambda)u}{\lambda} \right) dt, \]

\[ L_2(u, \delta, \gamma) = 1 + \frac{\Pr[A < R_1 < B]}{\lambda} \int_{-h}^{h} L_2(s, \delta, \gamma) f_{Y_t} \left( \frac{s - (1 - \lambda)u}{\lambda} \right) ds \]
\[ + 2 \frac{\Pr[A < R_1 < B]}{\lambda} \int_{-h}^{h} L(s, \delta, \gamma) f_{Y_t} \left( \frac{s - (1 - \lambda)u}{\lambda} \right) ds. \]
ARL and SMRL

\[ ARL_E(\delta, \gamma) = L(0, \delta, \gamma) \]
\[ SMRL_E(\delta, \gamma) = L_2(0, \delta, \gamma) \]
\[ SDRL_E(\delta, \gamma) = \left( SM_E(\delta, \gamma) - ARL_E(\delta, \gamma) \right)^{1/2}. \]
ARL and SMRL for (EWMA, $S^2$) chart with estimated parameters

We first develop conditional ARL and SMRL equations, given values of $W_0=w_0$ and $Z_0=Z_0$.

\[
L(u, \delta, \gamma \mid w_0, z_0) = 1 + \frac{\Pr[A < R_1 < B]}{\lambda} \int_{-h}^{h} L(t, \delta, \gamma \mid w_0, z_0) g_y \left( \frac{t-(1-\lambda)u}{\lambda} \right) \mid w_0, z_0 \, dt.
\]

\[
L_2(u, \delta, \gamma \mid w_0, z_0) = 1 + \frac{\Pr[A < R_1 < B]}{\lambda} \int_{-h}^{h} L_2(s, \delta, \gamma \mid w_0, z_0) g_y \left( \frac{s-(1-\lambda)u}{\lambda} \right) \mid w_0, z_0 \, ds
\]

\[
+ \frac{2 \Pr[A < R_1 < B]}{\lambda} \int_{-h}^{h} L(s, \delta, \gamma \mid w_0, z_0) g_y \left( \frac{s-(1-\lambda)u}{\lambda} \right) \mid w_0, z_0 \, ds.
\]
Unconditional Equations

\[ L(u, \delta, \gamma) = \int_{-\infty}^{\infty} \int_{0}^{\infty} L(u, \delta, \gamma | w_0, z_0)f_w(w_0)\phi(z_0)dw_0dz_0, \text{ and} \]

\[ L_2(u, \delta, \gamma) = \int_{-\infty}^{\infty} \int_{0}^{\infty} L_2(u, \delta, \gamma | w_0, z_0)f_w(w_0)\phi(z_0)dw_0dz_0. \]
$ARL_{EP} (\delta, \gamma) = L(0, \delta, \gamma)$

$SMRL_{EP} (\delta, \gamma) = L_2(0, \delta, \gamma)$

$SDRL_{EP} (\delta, \gamma) = \left( SM_{EP} (\delta, \gamma) - ARL_{EP} (\delta, \gamma) \right)^{1/2}.$
Numerical Results

Joint (Xbar, S²) chart

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Joint (EWMA, S²) chart λ=.05, h=0.3963

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### Joint (Xbar, S²) chart

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### Joint (CUSUM, S²) chart $k=.25$, $h=7.93$

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Conclusions

While the (EWMA,$S^2$) and (CUSUM,$S^2$) charts offer very attractive out-of-control ARLs when process parameters are known, they suffer from a very disheartening increase in false-alarm rate when process parameters are estimated from small or moderated numbers of retrospective samples.
Conclusions

The \((Xbar, S^2)\) chart with estimated parameters, while not signaling out-of-control as fast as the other alternatives, is found to be more robust in terms of false-alarm rate.

Users of these charts must balance the need for out-of-control detection, the risk of false-alarm, and the effect of the number of retrospective samples on parameter estimation.
Selected References

• Quesenberry (1993), JQT 36, pp. 95-108.
• Chen (1997), Statistica Sinica 7, pp. 401-407.
• Jones (2001), Technometrics 34, pp. 277-288.
• Jones (2004), JQT 36, pp. 95-108.